Abelian 3-cocycles from Quadratic Forms

Quinn's Formula

Kylan Schatz

Department of Mathematics
North Carolina State University

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What's the Big Idea?

For *2-groups* and *pointed braided fusion categories*, to understand their simple objects it is enough to understand group theory.

- Simple objects, associator, and braiding are given by $(G, A, [(\omega, c)])$.
- Even more simply, given by *pre-metric group* (G, A, q).
- It's easy to go extract quadratic form q from $[(\omega, c)]$. How to algebraically construct (ω, c) from q?

Definition (EGNO 2.1.1 [3])

A *monoidal category* is a category $\mathcal{C}$ equipped with:

- bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*,
- *associativity isomorphisms* $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$,
- and *unit* $(\mathbb{1}, \iota)$, $\iota : \mathbb{1} \otimes \mathbb{1} \xrightarrow{\sim} \mathbb{1}$,

satisfying the *pentagon axioms* and *unit axioms*.

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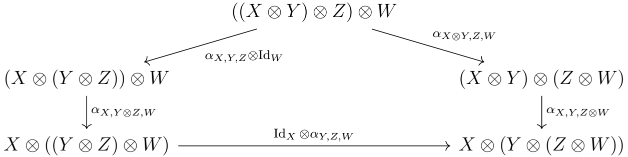
Pentagon Axioms

The following diagram commutes for all X, Y, Z, W:



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Unit Axioms

The functors:

$$L_{\mathbb{1}} : X \xrightarrow{\sim} \mathbb{1} \otimes X, \qquad \text{and} \qquad R_{\mathbb{1}} : X \xrightarrow{\sim} X \otimes \mathbb{1}$$

are autoequivalences of $\mathcal{C}$.

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Often, the theory of groups is easier to study than the theory of monoids. What type of category *categorifies* a group?

Definition (EGNO 2.10.1-2, 2.10.11)

An object X in monoidal category $\mathcal{C}$ has *left dual* X^* if there exist morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$, $\mathrm{coev}_X : \mathbb{1} \to X^* \otimes X$ such that the compositions:

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{Id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{Id}_X \otimes \mathrm{ev}_X} X,$$

$$X^* \xrightarrow{\mathrm{coev}_{X^*} \otimes \mathrm{Id}_{X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\alpha_{X^*,X,X^*}} X^* \otimes (X \otimes X^*) \xrightarrow{\mathrm{Id}_{X^*} \otimes \mathrm{ev}_{X^*}} X^*,$$

are identity morphisms.

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are identity morphisms.

A monoidal category $\mathcal{C}$ is *rigid* if every object has left and right duals.

Definition (EGNO 2.11.1, 2.11.4)

For $\mathcal{C}$ a rigid monoidal category, object X in $\mathcal{C}$ is *invertible* if $\mathrm{ev}_X, \mathrm{coev}_X$ are isomorphisms.

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Category of G-graded Vector Spaces

We consider the category $\Bbbk\text{-Vec}_G^\omega$ of G-graded $\Bbbk$-vectorspaces with associator data given by ω. Assume that $\Bbbk$ is algebraically closed.

Each object V has direct sum decomposition:

$$V = \bigoplus_{g \in G} V_g \cong \bigoplus_{g \in G} n_g \delta_g,$$

where δ_g is the unique simple object (up to isomorphism) in each graded component.

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and the associator is given:

$$\alpha_{g,h,k} : (\delta_g \otimes \delta_h) \otimes \delta_k \xrightarrow{\sim} \delta_g \otimes (\delta_h \otimes \delta_k),$$
$$\alpha_{g,h,k} = \omega(g,h,k)\,\mathrm{Id}_{\delta_{ghk}}.$$

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Tracing the pentagon axioms, we see that ω satisfies:

$$\omega(h,k,l)\omega(g,hk,l)\omega(g,h,k) = \omega(g,h,kl)\omega(gh,k,l),$$

meaning it must in fact be a *group 3-cocyle* on the group of isomorphism classes of simple objects. We denote:

$$\omega \in Z^3(G, \Bbbk^*).$$

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Can we use the theory of group cohomology to classify 2-groups up to monoidal equivalence?

A Classification

Theorem (Baez 4.2 [1], EGNO 2.11.5)

Monoidal equivalence classes of 2-groups are in bijections with triples (G, A, ω) where:

- *G is an Abelian group,*
- *A is a (in fact Abelian) G-module, and*
- *ω is an orbit in $H^3(G, A)$ under the action of $\mathrm{Out}(G)$*

Definition

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The following diagram commutes for all X, Y, Z:

$$\begin{array}{ccccc} & X \otimes (Y \otimes Z) & \xrightarrow{s_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X & \\ {\scriptstyle \alpha_{X,Y,Z}} \nearrow & & & & \searrow {\scriptstyle \alpha_{Y,Z,X}} \\ (X \otimes Y) \otimes Z & & & & Y \otimes (Z \otimes X) \\ {\scriptstyle s_{X,Y} \otimes \mathrm{Id}_Z} \searrow & & & & \nearrow {\scriptstyle \mathrm{Id}_Y \otimes s_{X,Z}} \\ & (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z) & \end{array}$$

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We say that braided 2-groups *categorify* the concept of Abelian groups; the isomorphism classes of objects form an Abelian group with operation $\otimes$.

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$$\delta_g \otimes \delta_h \cong \delta_{gh} = \delta_{hg} \cong \delta_h \otimes \delta_g,$$

defines a braiding on $\Bbbk\text{-Vec}_G^\omega$. Define the isomorphism:

$$s_{g,h} : \delta_g \otimes \delta_h \xrightarrow{\sim} \delta_h \otimes \delta_g,$$
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Tracing the hexagon axioms, we obtain:

$$\omega(h,k,g)c(g,h+k)\omega(g,h,k) = c(g,k)\omega(h,g,k)c(g,h),$$
$$\omega^{-1}(k,g,h)c(g+h,k)\omega^{-1}(g,h,k) = c(g,k)\omega^{-1}(g,k,h)c(h,k),$$

which make (ω, c) into an *Abelian 3-cocycle.*

A Classification

Proposition (EGNO 8.4.8)

braided monoidal equivalence classes of braided 2-groups are in bijection with triples $(G, A, (\omega, c))$ where:

- *G is an Abelian group,*
- *A is a (Abelian) G-module, and*
- *(ω, c) an orbit in $H^3_{ab}(G, A)$ under the action of $\mathrm{Aut}(G)$.*

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This is a nice classification, but cohomology is difficult to work with. Is there an easier representation still of each equivalence class of braided 2-groups?

Theorem (Eilenberg–MacLane, EGNO 8.4.9)

Let G, A Abelian groups and A a G-module. There is an isomorphism of groups:

$$\mathrm{tr} : H^3_{ab}(G, A) \xrightarrow{\sim} \mathrm{Quad}(G, A),$$
$$\mathrm{tr}[(\omega, c)](x) = q(x) \mapsto c(x, x),$$

where $\mathrm{Quad}(G, A)$ is the group of quadratic forms on G with coefficients in A.

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A priori, it seems like this map discards a lot of necessary information about the 3-cocycle $[(\omega, c)]$. To show that tr^{-1} is well-defined, we demonstrate (ω, c) for which $\mathrm{tr}[(\omega, c)] = q$.

Definition (Braunling 3.1, 3.2 [2])

Let $q \in \mathrm{Quad}(G, A)$ a quadratic form. Write its polarization form:

$$b(g,h) = \frac{q(g+h)}{q(g)q(h)}.$$

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We say triple (F_0, π, C) is a *pre-admissible presentation for* q if:

1. $\pi : F_0 \twoheadrightarrow G$ is a surjective group homomorphism,
2. $C : F_0 \otimes_{\mathbb{Z}} F_0 \to A$ is a bilinear form such that:

$$b(\pi x, \pi y) = C(x,y) + C(y,x),$$

3. C is alternating on $\ker \pi := F_1$, meaning:

$$C(x,x) = 0, \quad \forall x \in F_1.$$

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We say that (pre-)admissible presentation (F_0, π, C) is *optimal* if:

$$Q(x) = C(x,x), \quad \forall x \in F_0.$$

Definition

For a surjective group homomorphism $\pi : F_0 \to G$, we say that $\tilde{(\cdot)}$ is an *admissible lift* if:

$$\pi\tilde{x} = x, \quad \forall x \in G \qquad\qquad \text{and } \tilde{0} = 0.$$

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It is clear that, for a (pre-)admissible presentation (F_0, π, C), such an admissible lift always exists. We define a function:

$$L : G \times G \to F_0,$$
$$L(x, y) = \widetilde{(x + y)} - \tilde{x} - \tilde{y}$$

Existence Theorems

Proposition (Braunling 3.4)

For every (pre-)admissible presentation (F_0, π, C)*, there exists* C' *such that* (F_0, π, C') *is optimal (pre-)admissible.*

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Theorem (Braunling 6.1, 6.2)

Given (F_0, π, C) *an optimal admissible presentation for* $q \in \mathrm{Quad}(G, A)$ *and* $\tilde{(\cdot)}$ *an admissible lift, then:*

$$\omega(x, y, z) := -C(\tilde{x}, L(y, z)), \qquad c(x, y) := C(\tilde{x}, \tilde{y})$$

define an Abelian 3-cocycle such that $\mathrm{tr}[(\omega, c)] = q$.

Proposition (Braunling 4.5)

Suppose we have direct sum decomposition:

$$G = \left(\bigoplus_{\mathcal{I}} \mathbb{Z}/n_j\mathbb{Z}\right) \oplus \left(\bigoplus_{\mathcal{J}} \mathbb{Z}\right).$$

Then, an optimal admissible presentation for $q \in \mathrm{Quad}(G, A)$ *exists, and is of the form:*

$$0 \longrightarrow F_1 := \bigoplus\nolimits_{\mathcal{I}} \mathbb{Z} \hookrightarrow F_0 := \bigoplus\nolimits_{\mathcal{I} \coprod \mathcal{J}} \mathbb{Z} \xrightarrow{\pi} G \longrightarrow 0,$$

$$C(e_i, e_j) = \begin{cases} B(e_i, e_j) & i < j, \\ Q(e_i) & i = j, \\ 0 & i > j. \end{cases}$$

Why is this Sufficient?

Recall our definition of the polarization of quadratic form:

$$B(x,y) = Q(x+y) - Q(x) - Q(y)$$

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We recall that C is $\mathbb{Z}$-bilinear. To show optimality, we expand:

$$C(x, x) = C(\sum_i x_i e_i, \sum_i x_i e_i) = \sum_{i,j} x_i x_j C(e_i, e_j)$$
$$= \sum_{i<j} x_i x_j B(e_i, e_j) + \sum_i x_i^2 Q(e_i),$$

Why is this Sufficient?

Recall our definition of the polarization of quadratic form:

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as well as:

$$\begin{aligned} Q(x) &= Q(\sum_i x_i e_i) = B(x_{i_0} e_{i_0}, \sum_{i>i_0} x_i e_i) + Q(x_{i_0} e_{i_0}) + Q(\sum_{i>i_0} x_i e_i) \\ &= \cdots = \sum_{i<j} B(x_i e_i, x_j e_j) + \sum_i Q(x_i e_i) \\ &= \sum_{i<j} x_i x_j B(e_i, e_j) + \sum_i x_i^2 Q(e_i). \end{aligned}$$

Theorem (Braunling 7.1)

Let G, A Abelian groups, $q \in \mathrm{Quad}(G, A)$ and:

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Write $\{e_i : i \in \mathcal{I} \coprod \mathcal{J}\}$ the set of generators.

Define:

$$\sigma_{i,j} = \begin{cases} b(e_i, e_j) & i < j, \\ q(e_i) & i = j, \\ 0 & i > j. \end{cases}$$

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Write $\{e_i : i \in \mathcal{I} \coprod \mathcal{J}\}$ *the set of generators.*

Then,

$$\omega(x, y, z) = \sum_{\substack{i \in \mathcal{I} \\ y_j + z_j \geq n_j}} x_j n_j \sigma_{j,j}, \qquad c(x, y) = \sum_{i,j \in \mathcal{I} \coprod \mathcal{J}} x_i y_j \sigma_{i,j}.$$

define an Abelian 3-cocycle with $\mathrm{tr}[(\omega, c)] = q$.

Proof of Braunling 7.1 (Sketch).

By proposition (Br 4.5), we know an optimal admissible presentation of G exists. We need only choose an admissible lift. □

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We choose the standard lift:

$$\widetilde{(x + n_j\mathbb{Z})} = [x]_{n_j},$$

which is the remainder of x mod n_j as an integer. Thus,

$$L(x + n_j\mathbb{Z}, y + n_j\mathbb{Z}) = \begin{cases} -n_j & x + y \geq n_j, \\ 0 & x + y < n_j. \end{cases}$$

By (Br 6.2), we have that $\mathrm{tr}[(\omega, c)] = q$.

What about Fusion Categories?

We consider now a pointed, braided fusion category $\mathcal{C}$ over $\mathbb{C}$. The subclass of simple objects and isomorphisms between them give a *finite* Abelian group $[(\mathcal{C}_{simp}, \otimes)] := G$. We thus wish to pin down:

$$\text{tr} : H^3_{ab}(G, \mathbb{C}^*) \xrightarrow{\sim} \text{Quad}(G, \mathbb{C}^*).$$

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The inclusions on components give pullbacks:

$$\iota_k : \mathbb{Z}/n_k\mathbb{Z} \hookrightarrow G$$
$$\iota_k^* : \text{Quad}(G, \mathbb{C}^*) \twoheadrightarrow \text{Quad}(\mathbb{Z}/n_k\mathbb{Z}, \mathbb{C}^*) \cong \text{Hom}(\mathbb{Z}/(n_k^2, 2n_k)\mathbb{Z}, \mathbb{C}^*),$$

from a theorom by Whitehead. Thus, each $q(e_k)$ is $(n_j^2, 2n_j)$-torsion.

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Additionally, we find that since:

$$(n_k, n_l) = an_k + bn_l,$$

then each term $b(e_k, e_l)$ is (n_k, n_l)-torsion.

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- Choices of parameters:

$$q^{(k)} \in \{0, 1, \ldots, \gcd(n_k^2, 2n_k) - 1\} \text{ for all } k,$$
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- Abelian 3-cocycles given:

$$\omega(x, y, z) = \prod_k \exp\left(\frac{2\pi i q^{(k)}}{(n_k^2, 2n_k)} x_k([y_l]_{n_l} + [z_l]_{n_l} - [y_l + z_l]_{n_l})\right),$$
$$c(x, y) = \prod_{k<l} \exp\left(\frac{2\pi i b^{(k,l)}}{(n_k, n_l)} x_k y_l\right) \cdot \prod_k \exp\left(\frac{2\pi i q^{(k)}}{(n_k^2, 2n_k)} x_k^2\right)$$

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- Quadratic forms given:

$$b(e_k, e_l) = \exp\left(\frac{2\pi i b^{(k,l)}}{(n_k, n_l)}\right),$$
$$q(e_k) = \exp\left(\frac{2\pi i q^{(k)}}{(n_k^2, 2n_k)}\right).$$

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