Fusion Rules for $G^\times$-extensions

$N_{x,y}^{z} = \dim \mathrm{Hom}(z, x \otimes y)$

$u_{g;z}^{x,y} : \mathrm{Hom}(z, x \otimes y) \to \mathrm{Hom}(g_* z, g_* x \otimes g_* y)$

$V = \bigoplus V_g = \bigoplus \mathbb{C}[\mathrm{Fix}\, g],$

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For a fusion category $\mathcal{C}$, we have complete irreducibility into a direct sum of simples:

$$X \otimes Y = \bigoplus m_{x,y}^{i}\, e_i.$$

(think: vec sps, reps, ...)

By linearity of monoidal product, can consider simply $e_i \otimes e_j = \bigoplus N_{i,j}^{k}\, e_k$

$N_{i,j}^{k} = \dim \mathrm{Hom}(e_k, e_i \otimes e_j)$ fusion coefficient

Q: Is there a nice description for fusion coefficients? *

Recall: A short exact sequence is $(A, B, C \in \mathcal{C}$ Abelian$)$

$$0 \longrightarrow A \overset{\varphi}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} C \to 0$$

such that $\operatorname{Im}\varphi = \operatorname{Ker}\psi$.

We may call B an extension of A by C.

Def: A group action on monoidal category $G \curvearrowright \mathcal{C}$ is
[E] a monoidal functor (F, μ) such that

- $F \rightsquigarrow f \in \operatorname{Aut}_{gp} \operatorname{Irr}\mathcal{C}$, • $F(g) := (g_*, J_g)$ monoidal.

[E]

Def: Monoidal functor $(F, M) : \mathcal{C} \to \mathcal{C}'$,

$M_{x,y} : F(x) \otimes F(y) \xrightarrow{\sim} F(x \otimes y)$ such that:

$$
\begin{tikzcd}
(F(x) \otimes F(y)) \otimes F(z) \arrow[r] \arrow[d] & F(x) \otimes (F(y) \otimes F(z)) \arrow[d] \\
F(x \otimes y) \otimes F(z) \arrow[d] & F(x) \otimes F(y \otimes z) \arrow[d] \\
F((x \otimes y) \otimes z) \arrow[r] & F(x \otimes (y \otimes z))
\end{tikzcd}
$$

$\to$ $F(\alpha_{x,y,z}) \circ M_{x \otimes y, z} \circ (M_{x,y} \otimes \mathrm{Id}_{F(z)})$

$= M_{x, y \otimes z} \circ (\mathrm{Id}_{F(x)} \otimes M_{y,z}) \circ \alpha_{F(x), F(y), F(z)}.$

For a modular category, the data $G \curvearrowright \mathcal{C}$ is equivalently given by:

- Functor $F: \underline{G} \to \mathrm{Aut}_{\otimes} \mathrm{Irr}\, \mathcal{C}$
- Scalars $\mu^{x}_{f,g} : f_* g_* x \xrightarrow{\sim} fg_* x = \mu^{x}_{f,g}\, \mathrm{Id}$.
- Isomorphisms $U^{x,y}_{g;z} : \mathrm{Hom}(z, x \otimes y) \to \mathrm{Hom}(g_* z, g_* x \otimes g_* y)$

such that $\mu^{h_* x}_{f,g}\, \mu^{x}_{fg,h} = \mu^{x}_{g,h}\, \mu^{x}_{f,gh}$ and

$$\mu^{x}_{f,g}\, \mu^{y}_{f,g}\, U^{g_* x,\, g_* y}_{f;\, g_* z}\, U^{x,y}_{g;z} = \mu^{z}_{f,g}\, U^{x,y}_{fg;z}$$

For $\mathcal{C}$ a modular category and $G \curvearrowright \mathcal{C}$, define:

- $\mathrm{Fix}_g = \{ x \in \mathrm{Irr}\, \mathcal{C} : g_* x = x \}$, $(= \mathrm{Fix}_{g^{-1}})$
- $V_g = \mathbb{C}[\mathrm{Fix}_g]$, $V = \bigoplus_G V_g$

We call V a G-crossed extension of $\mathcal{C}$.

V is an associative algebra with:

"pointwise product"

- For $x \in \mathrm{Fix}_f$, $y \in \mathrm{Fix}_g$, $x \cdot y := \delta_{a=b}\, \mu^{x}_{f,g}\, x$

"convolution"

- For $x, y \in \mathrm{Fix}_g$, $x * y := \displaystyle\sum_{z \in \mathrm{Fix}_g} \frac{d_x d_y}{d_z} \mathrm{tr}(U^{x,y}_{g;z})\, z$.

☆ The study of these G-crossed extensions is parallel to the study of group characters.

Recall: the space of functions $\mathrm{Fun}(G) = \{f : G \to \mathbb{C}\}$ is an associative algebra with multiplication.

"convolution product"

We may define $f * g(x) = \frac{1}{|G|} \sum_{y \in G} f(xy^{-1})\, g(y)$

One can show that for $V = \bigoplus V_g$, both $\circ$ and $*$ are associative.

$e^2 = e$ $\qquad$ $e * d = \lambda e$

Recall: $(A, *)$ a semisimple commutative algebra

$\longleftrightarrow$ A has a basis of minimal idempotents.

Prop: $(\mathrm{Fun}(G), *) \cong \mathbb{C}[G]$ by $f \mapsto \sum_G f(g)\, g$

Corollary: $(\mathrm{ClassFun}(G), *) \cong Z(\mathbb{C}[G])$

$f(ghg^{-1}) = f(h) \ \forall g, h.$

Since $Z(\mathbb{C}[G])$ is commutative semisimple, we have a canonical basis of minimal idempotents for $\mathrm{ClassFun}(G)$, consisting of irreducible characters.

Theorem: $(V, *) = \bigoplus V_g$ is commutative semisimple.

Recall: • Braided monoidal equivalence classes of pointed, braided fusion categories given by

$$(A, [(\omega, c)]) \longleftrightarrow (A, q).$$

• (A, q) modular $\longleftrightarrow$ q nondegenerate.

Claim: The data of $G \curvearrowright \mathrm{Vec}(A, q)$ is given

• $F : G \to \mathrm{Aut}_{gp}(A)$, $g \mapsto g_*$,

• $f_* g_* x \xrightarrow{\sim} (fg)_* x := \mu^{x}_{f,g} \, \mathrm{Id}$,

• $g_* x \otimes g_* y \xrightarrow{\sim} g_*(x \otimes y) := \beta^{x,y}_{g} \, \mathrm{Id}$

+ coherences

"Proof:"

Coherences ↓

- $$\begin{tikzcd} (f_*g_*)h_*x \arrow[d] & = & f_*(g_*h_*)x \arrow[d] \\ f_{g*}h_*x \arrow[d] & & f_*gh_*x \arrow[d] \\ (fg)h_*x & = & f(gh)_*x \end{tikzcd}$$

$\longrightarrow$ • $\mu^{x}_{fg,h}\,\mu^{h_*x}_{f,g} = \mu^{x}_{f,gh}\,\mu^{x}_{g,h}$

μ^x 2-cocycle

- (g_*, J_g) monoidal $\longrightarrow$ • $\omega^{x,y,z}\,\beta_g^{xy,z}\,\beta_g^{x,y} = \beta_g^{x,yz}\,\beta_g^{y,z}\,\omega^{g_*x,g_*y,g_*z}$

β_g 2-coc

- $$\begin{tikzcd} f_*g_*x \otimes f_*g_*y \arrow[r] \arrow[d] & f_*(g_*x \otimes g_*y) \arrow[d] \\ f_{g*}x \otimes f_{g*}y \arrow[d] & f_*g_*(x\otimes y) \arrow[dl] \\ f_{g*}(x\otimes y) & \end{tikzcd}$$

$\longrightarrow$ • $\mu^{x\otimes y}_{f,g}\,\beta_g^{x,y}\,\beta_f^{g_*x,g_*y} = \beta_{fg}^{x,y}\,\mu^{x}_{f,g}\,\mu^{y}_{f,g}$

- $$\begin{tikzcd} g_*x \otimes g_*y \arrow[r] \arrow[d] & g_*y \otimes g_*x \arrow[d] \\ g_*(x\otimes y) \arrow[r] & g_*(y\otimes x) \end{tikzcd}$$

$\longrightarrow$ • $\beta_g^{y,x}\,c^{g_*x,g_*y} = c^{x,y}\,\beta_g^{x,y}$

In particular, when $G \to \mathrm{Aut}_{gp}(A)$ trivial,

$$G \curvearrowright \mathrm{Vec}(A, \alpha) \longleftrightarrow \begin{array}{l} \mu : A \to Z^2(G, \mathbb{C}^\times), \\ \beta : G \to Z^2(A, \mathbb{C}^\times) \end{array}$$

Additionally, β symmetric $\to$ trivializable.

Finally,

$$\begin{tikzcd}
z \arrow[r] \arrow[d, Rightarrow, "g_*"] & x \otimes y \\
g_* z \arrow[r] \arrow[dr, dashed, "U_{g;z}^{x,y}"'] & g_*(x \otimes y) \\
 & g_* x \otimes g_* y \arrow[u, "\beta_g^{x,y}"']
\end{tikzcd}$$

commutative + Schur's Lemma $\to \mathrm{tr}\, U_{g;z}^{x,y} = \delta_{x \otimes y \cong z}$.

Example: Consider $\mathbb{Z}_n \curvearrowright \mathrm{Vec}(\mathbb{Z}_m, q)$.

$[\omega] \to$ Prop: $H^{\cancel{*}}_{n}(\mathbb{Z}_m, \mathbb{C}^\times) \cong \begin{cases} \mathbb{C}^*, & n=0 \\ 0, & n \geq 1 \text{ odd} \\ \mathbb{Z}_m, & n \geq 2 \text{ even.} \end{cases}$ $\quad \mathbb{Z}_m \quad 0 \quad H^n$

Up to cohomology, μ, β both trivial.

Claim: $\mathrm{Idemp} = \frac{1}{\mathbb{Z}n}\left\{ \sum_{j=0}^{m-1} \zeta_m^{ij} \, \delta_j^g : 0 \leq i < m \right\} := \{e_i^g\}$. $\leftrightarrow \mathbb{Z}_m$ irreps.

$\zeta_m = e^{2\pi i/m}$

Additionally, $e_i^g \cdot e_j^h = \sum_{k,l=0}^{m-1} \zeta_m^{ik+jl} \, \underline{\delta_k^g \cdot \delta_l^h} = \delta_{k \simeq l}^{g+h}$

$$= \sum_{k=0}^{m-1} {}^{(i+j)k} \, \delta_k^{g+h} = e_{i+j}^{g+h}$$

$(g,k) + (h,l) = (g+h, k+l)$.

which recovers the fusion ring structure $\mathbb{Z}_n \times \mathbb{Z}_m$.

[J] Theorem: $(V, *)$ is a semisimple commutative algebra with basis of minimal idempotents $\{e_i^g\}$.

Additionally, for $d_i := \dim_q e_i^g$,

- $$e_i^g \cdot e_j^h = \sum_{\mathrm{Fix}_{gh}} \frac{d_i d_j}{d_k} N_{ij}^k e_k^{gh}$$ and

$$e_k^{gh} * (e_i^g \cdot e_j^h) = \frac{d_i d_j}{d_k} N_{ij}^k e_k^{gh}$$

★ Combining mul and convolution recovers (multiples of) the fusion coefficients!

Thm: $\dim \mathrm{Hom}(V_k, V_i \otimes V_j) = \langle \chi_k, \chi_i \cdot \chi_j \rangle$

θ_i

Q: How do we eliminate quantum dimension?

Claim: $(V, *)$ semisimple $\to$ there exists e_0 which is identity wrt. convolution. For every e_i^*, there exists e_i^* with $e_i^* \cdot e_i = \lambda e_0$.

Then, $e_0 * (e_i^* \cdot e_i) = \lambda \frac{d_{i^*} d_i}{d_0} N_{i,i^*}^{0} {}^{e_0} = \lambda d_i^2 e_0$

Example: For $\mathbb{Z}_n \curvearrowright \mathrm{Vec}(\mathbb{Z}_n, q)$, $e_0 = e_0^0$,

$e_i^{g*} = e_{-i}^{g^{-1}}$, $e_0^0 * (e_i^g \cdot e_i^{g*}) = e_0^0$

$\to \dim_i^2 = 1 \to \dim_i = 1$ (unitary).

$\to N_{ij}^k = \delta_{i+j=k}$.

Example: Consider $\mathbb{Z}_n \times \mathbb{Z}_m \circlearrowright \mathrm{Vec}(\mathbb{Z}_{(n,m)}, q)$

$\mathbb{Z}_n \times \mathbb{Z}_m \to \mathrm{Aut}(\mathbb{Z}_{(n,m)})$ triv.

Theorem: $H^2(\mathbb{Z}_n \times \mathbb{Z}_m, \mathbb{C}^\times) \cong \mathbb{Z}_{(n,m)}$ with

$b \mapsto [\Phi_b]$, $\quad \Phi_b(e_1^h e_2^t, e_1^k e_2^s) = \zeta_{(n,m)}^{b t k}$

Then,

$$\begin{tikzcd} \mathbb{Z}_{(n,m)} \arrow[r, "\mu"] \arrow[dr, dashed, "\times q"'] & Z^2(\mathbb{Z}_n \times \mathbb{Z}_m, \mathbb{C}^\times) \arrow[d] \\ & H^2(\mathbb{Z}_n \times \mathbb{Z}_m, \mathbb{C}^\times) \cong \mathbb{Z}_{(n,m)} \end{tikzcd}$$

Claim: $\mathrm{Idemp} = \coprod_{\mathbb{Z}_n \times \mathbb{Z}_m} \left\{ \sum_{j=0}^{(n,m)-1} \zeta_{(n,m)}^{ij} \delta_j^{h,t} : 0 \le i < (n,m) \right\} := \{ e_i^{h,t} \}$. $\leftrightarrow \mathbb{Z}_{(n,m)}$ irreps

$$e_i^{h,t} \cdot e_j^{k,s} = \cdots = \frac{1}{(n,m)} \sum_{a=0}^{(n,m)-1} \left(\sum_{b=0}^{(n,m)-1} \zeta_{(n,m)}^{b(i+j+qtk-a)} \right) e_a^{h+k, t+s}$$

$$= e_{i+j+qtk}^{h+k, t+s}$$

Then, $e_0^{0,0}$ is convolution identity, $e_i^{n,t\,*} = e_{-i+gtn}^{-n,-t}$,

$d_i = 1$ so $N_{ij}^{l} = \delta_{i+j+gtk=l}$

The last example recovers the fusion ring (extension):

$$0 \to \mathbb{Z}_{(n,m)} \to \mathbb{Z}_{(n,m)} \rtimes_{\omega_g} (\mathbb{Z}_n \times \mathbb{Z}_m) \to \mathbb{Z}_n \times \mathbb{Z}_m \to 0$$

$[\omega_g] \in H^2(\mathbb{Z}_n \times \mathbb{Z}_m, \mathbb{Z}_{(n,m)})$, $\omega_g(e_1^n e_2^t, e_1^k e_2^s) = gtk.$

Extension of $\mathbb{Z}_{(n,m)}$ by $\mathbb{Z}_n \times \mathbb{Z}_m$

$\longleftrightarrow$ $(\mathbb{Z}_n \times \mathbb{Z}_m)$-crossed extensions of $\mathrm{Vec}(\mathbb{Z}_{(n,m)}, q)$

with trivial $G \to \mathrm{Aut}_\otimes \mathrm{Irr}\, \mathcal{C}$.

Future work:

- $\mathbb{Z}_{p-1} \curvearrowright \mathrm{Vec}(\mathbb{Z}_p, \alpha)$ with nontrivial automorphism.
- $\mathbb{Z}_n \times \mathbb{Z}_m \curvearrowright \mathrm{Vec}(\mathbb{Z}_{(n,m)}, \alpha)$ nontrivial
- ...
- $\exists$? general formula $K \trianglelefteq G \curvearrowright \mathrm{Vec}(A, \alpha)$?
- For $\mathrm{Vec}(A, \alpha)$ non-modular, what do c_{ij}^k represent?
  (non-integer)

Questions?

References: [W] Weibel, C. Group Homology and Cohomology. Chapter 6.

[E] Etingof et al. Tensor Categories.

[J] Jones, C. Computing Fusion Rules for Spherical G-Crossed Extensions of Fusion Categories. arXiv: 1909.02816.