Extension Theory for Categories

Kylan Schatz (they/them), 8/2/2023

kaschatz@ncsu.edu

Preliminaries

- A *group* is a set (G) together with binary operation $(*)$, such that
  - There exists identity: $\exists 1 \in G$, such that $1g = g1 = g, \forall g \in G$,
  - There exist inverses: $\forall g \in G, \exists!\, g^{-1} \ni gg^{-1} = g^{-1}g = 1$,
  - The product is associative: $(g * h) * k = g * (h * k)$.
- A group is *Abelian* if it is commutative: $gh = hg$.
- Example: The integers with addition $(\mathbb{Z}, +)$.
  - Sub-example: The integers mod n with addition $(\mathbb{Z}_n, +)$.
- Example: Complex numbers with multiplication $(\mathbb{C} \setminus \{0\}, *)$.

Preliminaries

- A *short exact sequence* is a sequence of Abelian groups:
  - $0 \to A \overset{\varphi}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} C \to 0$, such that $\mathrm{Im}(\phi) = \ker(\psi)$.
  - We say that B *is an extension of* C *by* A.
- Example: The *direct product* $G \times H$ is an extension of H by G and conversely.
- Example: The *semidirect product* $G \rtimes H$ is an extension of H by G.
  - $G \cong G \rtimes \{1\}$ is an injection and $(g, h) \mapsto h$ is a surjection.

Preliminaries

- A *ring* is a set (R) together with two binary ops $(+,*)$, such that
  - R is an additive abelian group: $(R,+)$ Abelian,
  - Multiplication distributes over addition: $r*(s+t) = r*s + r*t$.
- A ring is called *unital* if $(R,*)$ is a monoid: $\exists!\, 1 \in R$, s.t. $1r = r1 = r$.
- Example: The *group ring* $\mathbb{Z}[G] = \mathbb{Z}G = \text{span}_{\mathbb{Z}}\{\text{g} \in G\}$, with ops
  - Addition: $\sum a_g g + \sum b_g g = \sum (a_g + b_g) g$,
  - Multiplication: $\left(\sum a_g g\right) * \left(\sum b_g g\right) = \sum a_g b_h gh$.

Preliminaries

- For G a group, a *G-module M* is an abelian group together with group homomorphism $\rho: G \to \mathrm{Aut}(M)$ for which:
  - $\rho(g)(m_1 + m_2) = \rho(g)m_1 + \rho(g)m_2$.
- Example: The *trivial module A*: $\rho(g)a = a$.
- Example: The *left-regular module G*: $\rho(g)(h + k) = g + h + k$.
- Example: The *free* $(\mathbb{Z})$*G-module generated by M:*
  - The group $(\mathbb{Z}M, +)$ with action $\rho(g)(\sum b_m m) = \sum b_m \rho(g)(m)$.
- Remark: The study of G-modules gives examples of *group extensions*.

Question

- What happens if we relax some of the axioms of groups?
- What if “equals” is not really equality?
  - What if we replace every “equals” with “isomorphic to?”

Categories

- A *category* $\mathcal{C}$ is a *class* $Ob(\mathcal{C})$ together with *morphisms* $\mathcal{C}(A, B)$ s.t.:
  - There exist identities: $\mathrm{id_A} \in \mathcal{C}(A, A)$,
  - Composition is associative: $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C) \Rightarrow gf \in \mathcal{C}(A, C)$.
- Example: (Vertical) categorification of a set X:
  - Construct a category $\mathcal{X}$ with $\mathrm{Ob}(\mathcal{X}) = X$ and $\mathcal{X}(A, B) = \delta_{A=B}\mathrm{id}$.
- Example: (Horizontal) categorification of a group
  - For a group G, we may construct a category $\mathcal{G}$ with $\mathrm{Ob}(\mathcal{G}) = \{*\}$ and morphisms $\mathcal{G}(*,*) = G$ are labeled by elements of G.

Categories

- An *Abelian category* is a category $\mathcal{C}$ together with bifunctor $\bigoplus: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *direct sum*, which admits kernels and cokernels.

- A *monoidal category* is an Abelian category together with
  - Bilinear bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product*
  - Isomorphisms $\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ satisfying the *pentagon equations,*
  - Unit object $(\mathbf{1}, \iota)$, $\mathbf{1} \in \mathcal{C}$ satisfying the *triangle equations*.

- Example: The category $\mathbb{k} - \mathsf{Vec}$ of vector spaces over $\mathbb{k}$.
  - The direct sum comes from concatenation of ordered bases.
  - The *associator* identifies $(V \otimes U) \otimes W \xrightarrow{\sim} V \otimes (U \otimes W)$. **This is not equality!**

Categories

- A monoidal category is called *strict* if α is equality.
- A monoidal category is *pointed* if each object has a tensor inverse.
- A monoidal category is *braided* if it has:
  - Braiding isomorphisms $S = \left\{S_{x,y}: X \otimes Y \xrightarrow{\sim} Y \otimes X\right\}$ satisfying the *hexagon equations*.
- A monoidal category is *fusion* if there are finitely many simple objects.

Categories

- Remark: Monoidal categories are the *(vertical) categorification* of monoids.
  - Pointed categories are the *(vertical) categorification* of groups,
  - Braided categories *categorify* Abelian groups,
  - Fusion categories *categorify* finite groups.
- Important example: $\mathrm{Vec}\big(G,(\omega,c)\big)$ is a pointed braided fusion category with:
  - Simple objects labeled by group elements δ_g, and $\delta_g \otimes \delta_h = \delta_{gh}$,
  - Associator $\alpha_{\delta_x,\delta_y,\delta_z}\colon \delta_x \otimes (\delta_y \otimes \delta_z) \xrightarrow{\sim} (\delta_x \otimes \delta_y) \otimes \delta_z = \omega(x,y,z)$,
  - Braiding $S_{\delta_x,\delta_y}\colon \delta_x \otimes \delta_y \xrightarrow{\sim} \delta_y \otimes \delta_x = c(x,y)$.

What's the Big Idea?

- There is the notion of a *group action on a category*.
- There is an analogue to *group extensions* using group actions on categories.
- Oftentimes, it is enough to describe the resulting category with group theoretical and cohomology data.

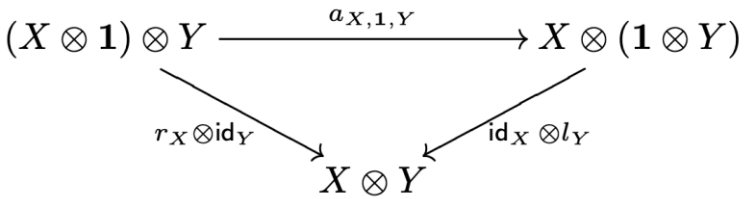
What's the Big Idea?

- Example: Trivial $\mathbb{Z}_n \times \mathbb{Z}_m \curvearrowright \mathtt{Vec}\left(\mathbb{Z}_{(n,m)}, (\omega, c)\right)$ results in:
  - Extension $\mathbb{Z}_{(n,m)} \hookrightarrow \mathbb{Z}_{(n,m)} \rtimes_{\varphi} (\mathbb{Z}_n \times \mathbb{Z}_m) \twoheadrightarrow \mathbb{Z}_n \times \mathbb{Z}_m$, where $\varphi\big((h,t),(k,l)\big) = gtk$.
  - Remark: Looks like a semidirect product.
- Example: $\mathbb{Z}_2 \curvearrowright \mathtt{Vec}(A, (\omega, c))$ by inverses, A finite abelian results in:
  - Extension $A \hookrightarrow A \rtimes_{\psi} \left(\frac{A}{A/K}\right) \twoheadrightarrow \left(\frac{A}{A/K}\right)$, where $\psi\left(\overline{b_1} + \frac{A}{K}, \overline{b_2} + \frac{A}{K}\right) = \overline{b_1} + \overline{b_2} + \bar{h} + \frac{A}{K}$.
  - Remark: Looks like *Tambara-Yamagami* category.

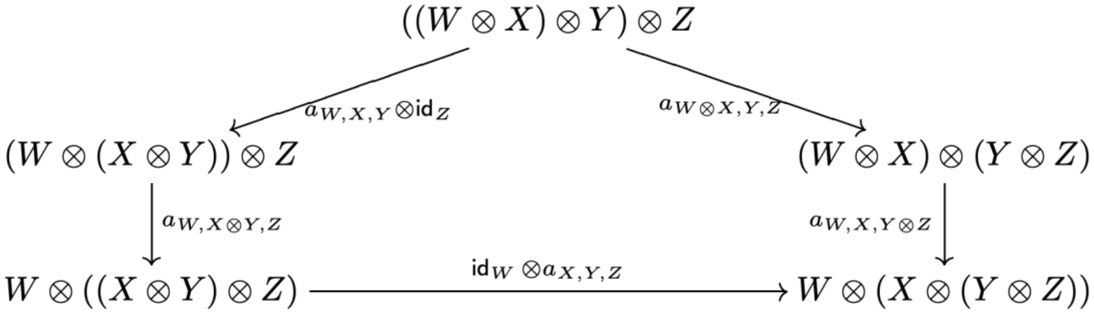
Closing

- I was "scooped" on this project :/ but the tools are still useful!
- Research moving in to more braided-category theory direction.
- Upcoming project to incorporate *operator theory* (an analysis topic).

Questions?



$$\beta_g^{k,l}\beta_g^{j,kl}\omega^{g_*j,g_*k,g_*l} = \omega^{j,k,l}\beta_g^{j,k}\beta_g^{jk,l},$$
$$\beta_g^{l,k}c^{g_*k,g_*l} = c^{k,l}\beta_g^{k,l},$$
$$\mu_{fg,h}^{k}\mu_{f,g}^{h_*k} = \mu_{g,h}^{k}\mu_{f,gh}^{k},$$
$$\mu_{g,h}^{kl}\beta_h^{k,l}\beta_g^{h_*k,h_*l} = \beta_{gh}^{k,l}\mu_{g,h}^{k}\mu_{g,h}^{l}.$$



$$\dim \mathrm{Hom}_\times(V_{b_3}^0, V_{b_1}^0 \otimes V_{b_2}^0) = \delta_{b_1+b_2=b_3},$$
$$\dim \mathrm{Hom}_\times(V_{\bar{b}_2}^1, V_b^0 \otimes V_{\bar{b}_1}^1) = \dim \mathrm{Hom}_\times(V_{\bar{b}_2}^1, V_{\bar{b}_1}^1 \otimes V_b^0) = \delta_{\bar{b}_2=b+\bar{b}_1},$$
$$\dim \mathrm{Hom}_\times(V_b^0, V_{\bar{b}_1}^1 \otimes V_{\bar{b}_2}^2) = \delta_{b\in(\bar{b}_1+\bar{b}_2+h)+G\backslash\bar{G}}.$$