A Hierarchy of Symmetry for QCA

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Abstract Spin Systems

An *abstract spin system* (ASC) consists of:

- A *net* of algebras $2^{\mathbb{Z}} \to C^*(\mathbb{C})$ such that:
  i. If $I \subseteq J$, then $A_I \subseteq A_J$ unital, and
  ii. If $I \cap J = \emptyset$, then $[A_I, A_j] = 0$.

Given ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ such that $\bigcup I_i = \mathbb{Z}$,

$A = \overline{\bigcup A_{I_i}}^{\|\cdot\|}$ is a (quasi-local) AF-algebra, called the *inductive limit*.

Abstract Spin Systems

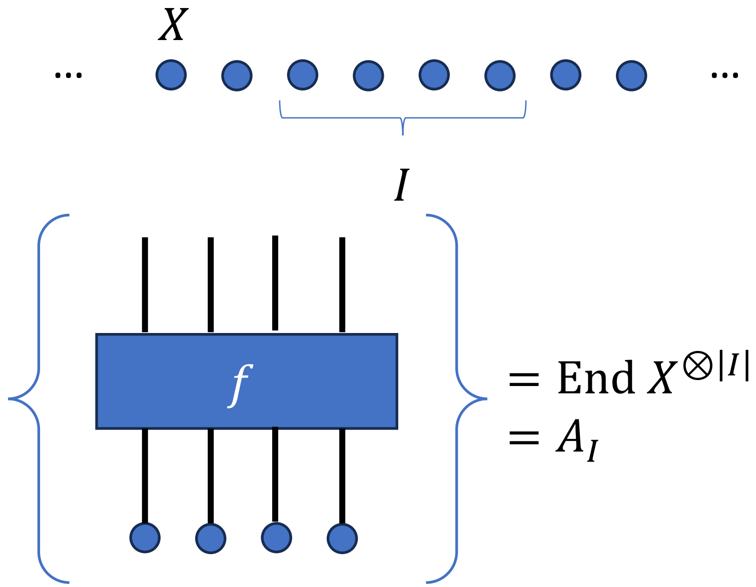
Suppose you are given (quasi-local) AF-algebra A.

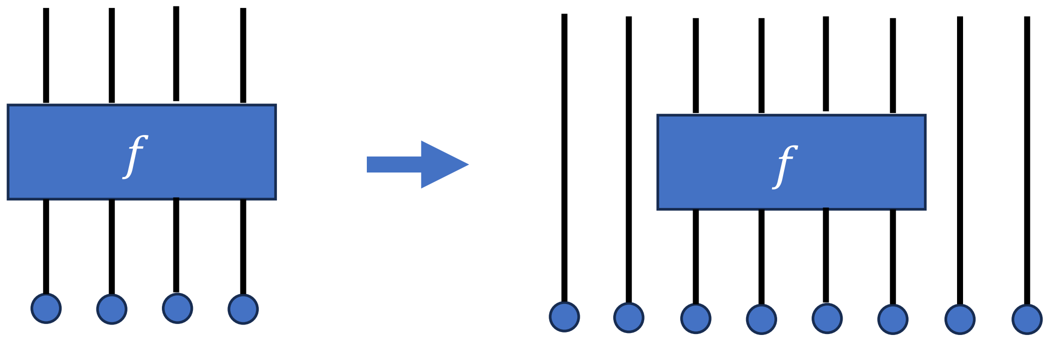
If you can construct (poset) morphism $2^{\mathbb{Z}} \to \{A_I \subset A \text{ unital} : A_I \text{ fd}\}$ (satisfying i. ii.) then you can recover an abstract spin system!

Fusion Categorical Spin Systems

Let C a (multi)fusion category and $X \in \mathrm{Ob}\ C$. We define:

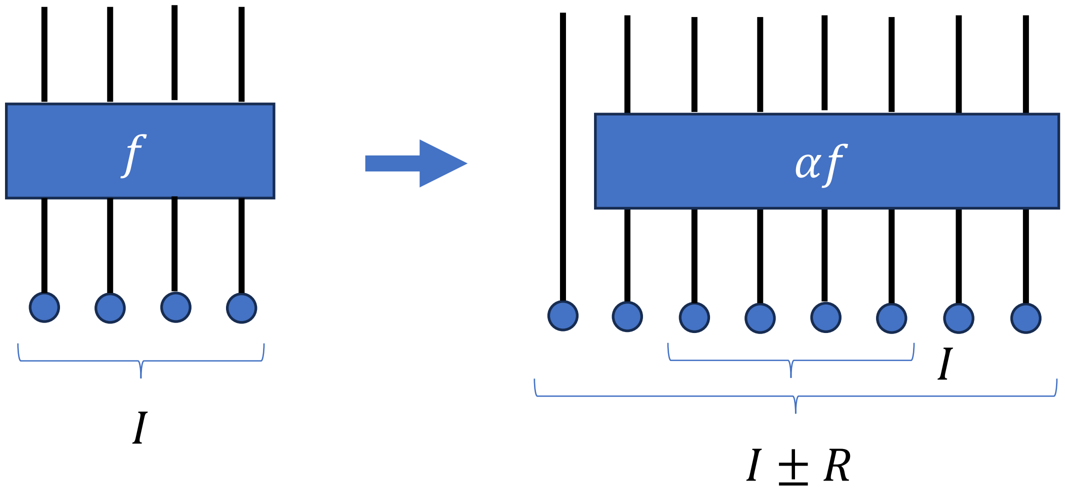
The resulting *fusion spin system* (FSC) is called $A(C, X)$.





Quantum Cellular Automata

A *quantum cellular automata* (QCA) is a bounded spread isomorphism:
$\alpha \in \mathrm{Aut}\, A$ such that $\forall I$, there exists $R > 0$ such that $\alpha A_I \subseteq A_{I \pm R}$.



Quantum Cellular Automata

Fusion spin systems correspond to *equivariantization* of "ordinary" spin systems:

Given (multi)fusion D, $X \in \mathrm{Ob}\, D$, and action $D \curvearrowright A(\mathrm{Vec}, \mathrm{End}\, X)$, then:

$$A(\mathrm{Vec}, \mathrm{End}\, X)^D \cong A(D^{op}, X).$$

Additionally, $\alpha \in \mathrm{QCA}\, A(D^{op}, X)$ extends to $\tilde{\alpha} \in \mathrm{QCA}\, A(\mathrm{Vec}, \mathrm{End}\, X)$ iff $\mathrm{DHR}(\alpha)L \cong L$ as algebra objects.

Extensions of Spin Systems

Given $F: C \to D$ a *dominant* tensor functor, then we have lift:

$$\tilde{F}: A(C, X) \to A(D, FX).$$

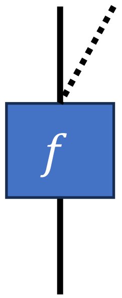
If $F: C \to D$ *dominant*, then $\exists!\, A \in Z(C)$ an algebra object such that $D \cong C_A$ (category of *right A-modules*) as tensor categories.

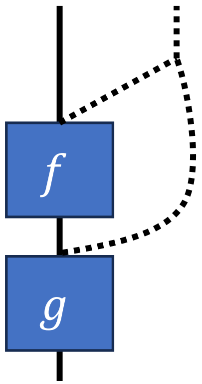
Additionally, $F: C \to D \cong \iota: C \hookrightarrow C_A$, which sends $X \mapsto X \otimes A$.

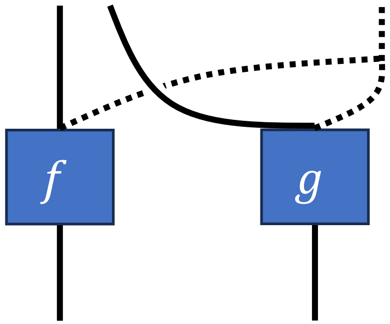
Right Modules

Define the category $\widehat{C_A}$ with $\mathrm{Ob}\,\widehat{C_A} = \mathrm{Ob}\,C$,

- Morphisms $\widehat{C_A}(X, Y) = C(X, Y \otimes A)$
- Composition $f \circ g = (\mathrm{id} \otimes \mu) \circ (f \otimes \mathrm{id}) \circ g$
- Tensor $f \otimes g = (\mathrm{id} \otimes \mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \psi_Y \otimes \mathrm{id}) \circ (g \otimes f)$







Right Modules

The category $\widehat{C_A}$ is tensor, but we do not take simples to simples.

To get (multi)fusion, we must take *idempotent completion*.

Given category C, we define its *idempotent completion* C^0 with

- Objects $\mathrm{Ob}\, C^0 = \{(X, p)\}, X \in \mathrm{Ob}\, C, p^2 = p \in \mathrm{End}\, X$, and
- Morphisms $C^0\big((X, p), (Y, q)\big) = \{f \in C(X, Y) : fp = f = qf\}$

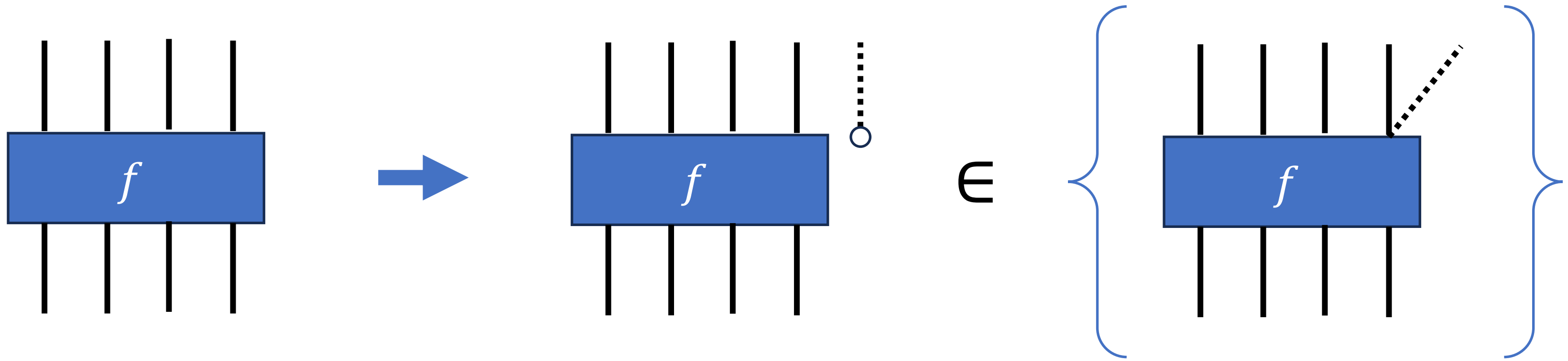
The category of *right A-modules* is $C_A = \widehat{C_A}^0$.

Extensions of Spin Systems

Given $F: C \to D$ a *dominant* tensor functor, then we have lift:

$$\tilde{F}: A(C, X) \to A(D, FX).$$

To map spin systems, we take $f \mapsto f \otimes \iota$:



Hypergroups

A *hypergroup* H (over $\mathbb{C}$) is a *convex set* with *unity*, associative *convex multiplication*, and *weak inverses*.

- There exists unique $e_0 \in H$ such that $x = e_0 x = x e_0$ for all $x \in H$.
- There exists a distinguished convex basis $\{e_i\} \subseteq H$, meaning $e_i e_j = \sum \lambda_{i,j}^k e_k$, where $\lambda_{i,j}^k \geq 0$ and $\sum \lambda_{i,j}^k = 1$.
- For each e_i, there is a unique $e_{\bar{\imath}}$ such that $\lambda_{i,\bar{\imath}}^0 \neq 0$.

Hypergroups

Example: The convex hull of $G \subseteq \mathbb{C}G$ is a hypergroup.

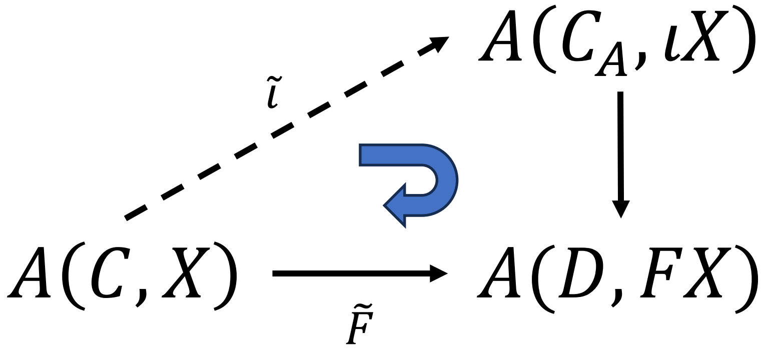
Example: $(\mathrm{End}\, A, *, \circ)$ for A a commutative algebra object is a hypergroup with basis of *convolution idempotents* called $\mathrm{HyperAut}\, A$.

A *hypergroup action* of H on commutative algebra object A is a *hypergroup homomorphism* $H \to \mathrm{HyperAut}\, A$.

Hypergroup Symmetries of QCA

For $F: C \to D$ a dominant tensor functor, we have commutative:

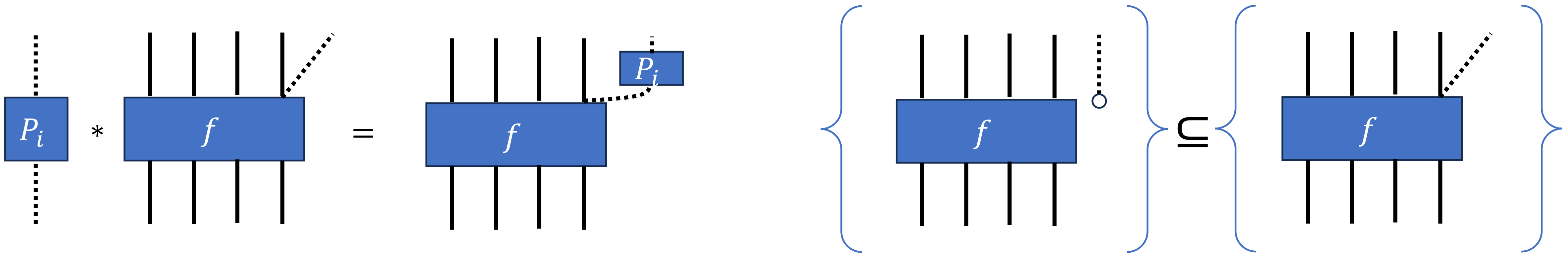
When does this diagram commute with hypergroup action $H \curvearrowright A$?



Hypergroup Symmetries of QCA

Recall: A commutative algebra object means $\mathrm{End}\, A$ has basis of convolution idempotents (projections) $\{P_i\}$.

Claim: $A(C_A, \iota X)^{\mathrm{HyperAut}\, A} = A(C, X)$, by



Questions?