The 2-Group of Lagrangian Algebra-Equivariant Autoequivalences

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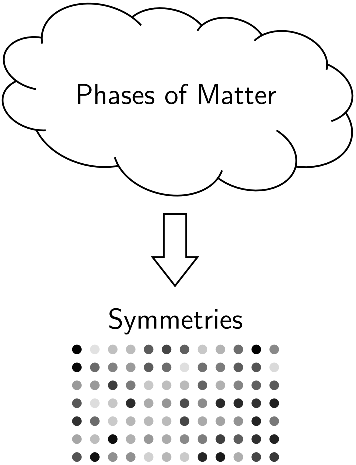
What's the Big Idea?



In physics, *orders* are used to study phases of matter.

- Certain symmetries are assigned to each phase.
- Breaking of symmetry can indicate a *change in phase*.
- Classically, order parameters are group symmetries.
- In *(2+1)D topological case*, order parameters are *MTCs*.

Topological Order

- *Topological phases of matter* are important in the study of *condensed matter physics*:
  - e.g. (topological) quantum computation, superconductors.
- A *(2+1)D topological order* is an assignment of a modular category to a (2+1)D *gapped* topological phase of matter.
- When the phase carries an additional group symmetry, one may construct a corresponding *symmetry enriched* order.
- (2+1)D *symmetry enriched topological orders (SETOs)* are G-crossed braided extensions of a modular category.

Anyon Condensation

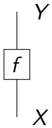
We are particularly interested in phase transition between gapped SETOs described by *anyon condensation*.

- Condensable anyons in a phase are described by *commutative algebra objects* in the associated MTC.
- Transitions across a gapped boundary to the trivial phase correspond to *Lagrangian algebra objects*.
- Obstructions to the spontaneous breaking of symmetry under anyon condensation are equivalent to *equivariant structure* on the condensed anyon.

Graphical Calculus for Monoidal Categories

We use the graphic calculus and the bottom-up / "optimistic" convention to represent the monoidal category $\mathcal{C}$:

$X \in \mathrm{Ob}(\mathcal{C}) \rightsquigarrow$ 

, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \rightsquigarrow$ 

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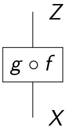
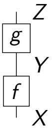
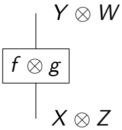
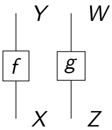


$=$ 

, and 

$=$ 

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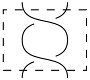
Modular Categories

Definition

A *modular* category is a ribbon fusion category for which the matrix:

$$S = \left(\mathrm{Tr}_{V_i \otimes V_j} \left(\quad \right) \right)_{i,j}$$



$c_{V_j,V_i} \circ c_{V_i,V_j}$

is invertible.

This is the 'correct' categorification of a finite group.

Drinfeld Centers

Definition ([EGNO16, Definition 7.13.1])

Associated to the fusion category $\mathcal{C}$ is its *Drinfeld center* $\mathcal{Z}(\mathcal{C})$, whose objects are tuples (X, ψ) where $X \in \mathrm{Ob}(\mathcal{C})$ and $\psi : X \otimes \_ \xrightarrow{\sim} \_ \otimes X$ is a *half-braiding*.

Remark ([BV13])

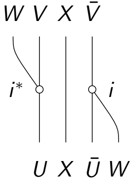
For $\mathcal{C}$ unitary fusion, the center $\mathcal{Z}(\mathcal{C})$ has the natural structure of a modular category with the braiding $C_{(X,\psi),(Y,\varphi)} = \psi_Y$.

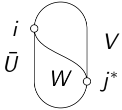
The Induction Functor

Definition ([KB10, Theorem 2.3], [BJ22, Section 3.1])

The *induction functor* $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is the (left) adjoint to the forgetful functor $\mathrm{Forg} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. We use the concrete model:

$$X \mapsto \left(\bigoplus_{U \in \operatorname{Irr} \mathcal{C}} U \otimes X \otimes \bar{U},\ \psi_{I(X)} \right), \quad f \mapsto \bigoplus_{U \in \operatorname{Irr} \mathcal{C}} \mathrm{id}_U \otimes f \otimes \mathrm{id}_{\bar{U}},$$

$$\psi_{I(X),W} = \bigoplus_{U,V \in \operatorname{Irr} \mathcal{C}} \sum_i \sqrt{d_U}\sqrt{d_V}$$ 

, $\quad \delta_{i,j} =$ 

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Recovering $\mathcal{C}$ from its center $\mathcal{Z}(\mathcal{C})$

For $\mathcal{C}$ unitary fusion, the object $L := I(\mathbb{1})$ has the structure of a connected, separable, commutative Frobenius algebra called the *canonical Lagrangian algebra*.

Proposition (Bruguiéres, Natale 2010)

$I_L : X \mapsto (I(X), \eta^I_{X,\mathbb{1}})$ is a tensor equivalence and the following commutes up to monoidal natural isomorphism:

$$\begin{array}{ccc} & & \mathcal{C} \\ & \overset{\text{Forg}}{\nearrow} & \big\downarrow I_L \\ \mathcal{Z}(\mathcal{C}) & \xrightarrow[\text{Free}]{} & \mathcal{Z}(\mathcal{C})_L \end{array} .$$

2-Groups

Definition ([EGNO16, Definition 2.11.4])

A *2-group (Gr-category, categorical group)* is a 2-category with one object whose morphisms are invertible up to 2-morphism and whose 2-morphisms are invertible.

Example

- Groups are 2-groups.
- Each monoidal category has an associated 2-group.
- The autoequivalences of a category form a 2-group.

The 2-Group $\underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A)$

Definition (Bischoff, Jones, Lu, Penneys 2019)

Let $\mathcal{C}$ a braided monoidal category, $A \in \mathrm{Ob}\,\mathcal{C}$ a commutative, separable algebra object. The categorical group $\underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A)$ has one object, whose

- morphisms are tuples $(\alpha, \eta^{\alpha}, \lambda^{\alpha})$ where
  - (α, η^{α}) is a braided monoidal autoequivalence and
  - $\lambda^{\alpha} : A \xrightarrow{\sim} \alpha(A)$ an algebra isomorphism, and
- 2-morphisms are monoidal natural transformations $\pi \in \mathrm{Hom}((\alpha, \eta^{\alpha}), (\beta, \eta^{\beta}))$ such that $\pi_A \circ \lambda^{\alpha} = \lambda^{\beta}$.

Symmetry Breaking

Big Idea: The 2-group $\underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A)$ describes equivariant structures on the algebra object A.

Theorem (Bischoff, Jones, Lu, Penneys 2019)

Obstructions to preservation of group symmetry under anyon condensation are equivalent to lifts:

$$\begin{array}{ccccc} & & \underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A) & \xrightarrow{\underline{F_A}} & \underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}_A^{loc}) \\ & \nearrow & \downarrow & & \\ \underline{G} & \longrightarrow & \underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}) & & \end{array} .$$

Classifying Obstructions

Question

Is there a characterization of the 2-group $\underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A)$?

Theorem (S 2024)

Let $\mathcal{D}$ unitary fusion and $L = I(\mathbb{1})$ its canonical Lagrangian algebra. Then, there is an equivalence of 2-groups:

$$\underline{I^L} : \underline{\mathrm{Aut}_{\otimes}}(\mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{Z}(\mathcal{D})|L).$$

Remark

This is a sort of 'bulk-boundary correspondence' for symmetries.

What Was Known

Remark ([JMNR21], [ENO10])

It is already known that there is group homomorphism:

$$I^L : \mathrm{Aut}_\otimes(\mathcal{D}) \to \mathrm{Aut}_\otimes^{br}(\mathcal{Z}(\mathcal{D})), \qquad \text{with}$$
$$\mathrm{Im}(I^L) = \mathrm{Stab}(L).$$

We provide a categorification and a factorization:

$$\begin{array}{ccccc} & & \mathrm{Aut}_\otimes^{br}(\mathcal{Z}(\mathcal{D})|L) & & \\ & \overset{I^L}{\nearrow} & \downarrow \mathrm{Forg} & & \\ \mathrm{Aut}_\otimes(\mathcal{D}) & \twoheadrightarrow & \mathrm{Stab}(L) & \hookrightarrow & \mathrm{Aut}_\otimes^{br}(\mathcal{Z}(\mathcal{D})) \end{array}$$

Outline of Proof

- The equivalence I_L [BN11] lifts to an equivalence of 2-groups.
- In their paper describing symmetry breaking from anyon condensation, [BJLP19] define a functor $\underline{F_L}$.
- We show the following diagram commutes (up to monoidal natural isomorphism) and that $\underline{F_L}$ is faithful.

$$\begin{array}{ccc}
\underline{\mathrm{Aut}}_{\otimes}(\mathcal{D}) & \xrightarrow{\underline{I^L}} & \underline{\mathrm{Aut}}_{\otimes}^{br}(\mathcal{Z}(\mathcal{D})|L) \\
& \underset{\underline{I_L}}{\searrow} & \downarrow \underline{F_L} \\
& & \underline{\mathrm{Aut}}_{\otimes}(\mathcal{Z}(\mathcal{D})_L)
\end{array}$$

Hypergroups

Definition

A *hypergroup* is a simplex $\mathsf{H} = \mathrm{conv}_{\mathbb{C}}\{\mathsf{e}_0, \ldots, \mathsf{e}_{\mathsf{n}-1}\}$ which is:

- a monoid with identity e_0,
- whose multiplication is H-linear, and
- for which each $e_\imath$ has a unique weak inverse $e_{\bar{\imath}}$.

Example

- $\mathrm{conv}_{\mathbb{C}}\{G\} \subseteq \mathbb{C}[G]$ is a hypergroup.
- characters of a finite group form a hypergroup.

Grothendieck Hypergroup

Example

Associated to the fusion ring C is the hypergroup:

$$\mathrm{K}_0(C) := \mathrm{conv}_{\mathbb{C}}\{d_X\, X\}_{X \in \mathcal{B}(C)}, \qquad d_X := \sqrt{c^1_{X,\bar{X}}}.$$

For fusion category $\mathcal{C}$ with associated fusion ring $\mathrm{Fus}\,\mathcal{C}$, we denote:

$$\mathrm{K}_0(\mathcal{C}) := \mathrm{K}_0(\mathrm{Fus}\,\mathcal{C})$$

HyperAutomorphisms

Proposition (Bischoff, Davydov 2020)

For commutative separable algebra object A in a unitary modular category, $\mathrm{End}(A)$ *is commutative semisimple with respect to the convolution product.*

Definition

HyperAut(A) is the hypergroup whose extreme points are minimal convolution idempotents of $\mathrm{End}(A)$. A *hypergroup action* of H on A is a morphism:

$$\mathrm{H} \to \mathrm{HyperAut(A)}.$$

Some 2-Subgroups

Definition

Suppose you have hypergroup action $\mathsf{H} \curvearrowright \mathsf{A}$. $\underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A, \mathsf{H})$ is the full 2-subgroup of $\underline{\mathrm{Aut}_{\otimes}^{br}}(\mathcal{C}|A)$ for which:

$$\lambda^{\alpha} \circ e_i = \alpha(e_i) \circ \lambda^{\alpha}, \qquad \forall e_i \in \mathsf{H}.$$

Definition

Let $\mathcal{E} \subseteq \mathcal{D}$. $\underline{\mathrm{Aut}_{\otimes}}(\mathcal{D}|_{\mathcal{E}})$ is the full 2-subgroup of $\underline{\mathrm{Aut}_{\otimes}}(\mathcal{D})$ where:

$$\alpha|_{\mathcal{E}} \cong \mathrm{Id}_{\mathcal{E}} \qquad \text{as } \textit{functors}.$$

A Second Classification

Corollary (S 2024)

Let $\mathcal{D}$ unitary fusion and $L = I(\mathbb{1})$ its canonical Lagrangian algebra, and $\mathcal{E} \subseteq \mathcal{D}$ full fusion. Then, there is an equivalence of 2-groups:

$$\underline{I^L_{\mathcal{E}}} : \underline{\mathrm{Aut}_{\otimes}}(\mathcal{D}|_{\mathcal{E}}) \xrightarrow{\sim} \underline{\mathrm{Aut}^{br}_{\otimes}}(\mathcal{Z}(\mathcal{D})|L, \mathrm{K}_0(\mathcal{E})).$$

Remark

This is an extension of our 'bulk-boundary correspondence of symmetries' to include matrix product operator symmetries.

Outline of Proof

- [BJ22] give a canonical action of $\mathcal{D}$ on $I(\mathbb{1}) \in \mathcal{Z}(\mathcal{D})$.
- We show the following diagram of groups commutes and argue about the images of subgroups:

$$\begin{array}{ccc} \mathrm{Aut}_{\otimes}(\mathcal{D}) & \xrightarrow{I^L} & \mathrm{Aut}_{\otimes}^{br}(\mathcal{Z}(\mathcal{D})|L) \\ \downarrow{\scriptstyle \mathrm{Forg}} & & \downarrow{\scriptstyle \mathrm{ad}} \\ \mathrm{Aut}\,\mathrm{Fus}\,\mathcal{D} & \xrightarrow[\mathrm{can}]{} & \mathrm{Aut}\,\mathrm{End}\,L \end{array},$$

- Since the desired 2-subgroups were full, we immediately obtain the categorified result.

Subcategories of $\mathrm{Vec}\,(G)$

Full fusion subcategories of $\mathrm{Vec}\,(G)$ correspond to subgroups $H \subseteq G$. Isomorphism classes of monoidal autoequivalences are given [NR14]:

$$\mathrm{Aut}_{\otimes}(\mathrm{Vec}\,G) \cong \mathrm{Aut}(G) \ltimes H^2(G), \\ [\alpha, \eta^\alpha] \mapsto (\mathrm{Forg}(\alpha), [\eta^\alpha])$$

Notice there is a canonical action of $\mathrm{Aut}(G)$ on G. We claim that:

$$\mathrm{Aut}_{\otimes}(\mathrm{Vec}\,G|_{\mathrm{Vec}\,H}) \cong \mathrm{Stab}(H) \ltimes H^2(G).$$

The Subcategories of $\mathrm{Vec}\,(S_3)$

Consider the symmetric group on 3 elements S_3 which has $H^2(S_3) = 0$ and $\mathrm{Aut}(S_3) = S_3$. Subgroups of S_3 are:

- 0 with $\mathrm{Stab}(0) = S_3$.
- $(12), (13), (23)$ with:

$$\mathrm{Stab}((12)) = \{123, 132\} \cong \mathbb{Z}_2,$$
$$\mathrm{Stab}((23)) = \{123, 321\} \cong \mathbb{Z}_2,$$
$$\mathrm{Stab}((13)) = \{123, 213\} \cong \mathbb{Z}_2.$$

- (123), with $\mathrm{Stab}((123)) = \{123, 231, 312\}$.
- S_3 with $\mathrm{Stab}(S_3) = \{123\}$.

The Subcategories of $\mathrm{Vec}\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)$

Consider the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which has $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_3$. Nontrivial subgroups of S_3 are:

- 0 with $\mathrm{Stab}(0) = S_3 \ltimes \mathbb{Z}_2 \cong S_3 \times \mathbb{Z}_2$,
- $\langle(1,0)\rangle, \langle(0,1)\rangle, \langle(1,1)\rangle$ are all normal with:

$$\begin{aligned}
\mathrm{Stab}((1,0)) &= \{123, 132\} \ltimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\
\mathrm{Stab}((0,1)) &= \{123, 321\} \ltimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\
\mathrm{Stab}((1,1)) &= \{123, 213\} \ltimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\end{aligned}$$

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathrm{Stab}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2$.

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